Journal of Statistical Physics, Vol. 25, No. 1, 1981

Piecewise Optimal Linearization Method for Nonlinear Stochastic Differential Equations

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Received January 13, 1980

A method for estimating the dynamical statistical properties of the solutions of nonlinear Langevin-type stochastic differential equations is presented. The nonlinear equation is linearized within a small interval of the independent variable and statistical properties are expressed analytically within the interval. The linearization procedure is optimal in the sense of the Chebyshev inequality. Long-term behavior of the solution process is obtained by appropriately matching the approximate solutions at the boundaries between intervals. The method is applied to a model nonlinear equation for which the exact time-dependent moments can be obtained by numerical methods. The calculations demonstrate that the method represents a significant improvement over the method of statistical linearization in time regimes far from equilibrium.

KEY WORDS: Nonlinear stochastic equations; time-dependent moments; statistical linearization; piecewise optimal linearization.

1. INTRODUCTION

The addition of random inhomogeneities to otherwise deterministic differential equations is a well-known device for formally reducing the number of degrees of freedom in many-body problems which arise in several areas of physics, chemistry, engineering, and biology. The Langevin equation for the velocity of a Brownian particle is, of course, the most celebrated example of this approach. In general, such stochastic differential equations arise when a large system is divided into a "relevant" part to be studied in detail and

Supported in part by the National Science Foundation under Grants CHE77-16307 and PHY76-04761.

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0022-4715/81/0500-0127$03.00/0

the remaining "environment" which is assumed to be described in terms of known statistical properties. For a few examples, the contraction of description and identification of statistical elements can be carried out exactly. However, in most problems the "stochastification"[1] of deterministic equations is performed in an ad hoc manner based on one's intuition regarding the underlying microscopic physics. In any case, there is a large and rapidly growing body of literature dealing with specific applications of this basic method.

Both linear and nonlinear stochastic differential equations play important roles in stochastic models of physical phenomena.[1–7] However, elementary discussions of stochastic differential equations concentrate on linear equations having constant coefficients and Gaussian, Markovian driving terms. While it is straightforward to extend the treatment to linear equations having non-Markovian and/or non-Gaussian inhomogeneities, the treatment of nonlinear equations is, in general, difficult. For linear equations, statistical properties of the solution process can be obtained from formal analytic solutions of the differential equation in terms of the random inhomogeneity.[2] Nonlinear equations which do not admit analytic solutions must generally be solved by approximate methods.[2] An essentially exact numerical method for linear or nonlinear equations of the type described above is sample function numerical integration in which the stochastic process is generated as a collection of its individual realizations. This method has been profitably applied to several problems in chemical physics, where it is known as the method of classical stochastic trajectories.[3,4] Although physically appealing, this approach often has the disadvantage of being computationally expensive especially when long-time integrations are necessary.[3] Therefore it is of practical importance to devise accurate and computationally efficient approximation methods for solving nonlinear stochastic differential equations. Many approximation methods have already been introduced for this type of equation.[1–3,7,9–11] A large number of these attempt to linearize the nonlinear differential equation and obtain the exact solution of the linearized problem. For example, standard

[2] Although it is possible to convert a nonlinear ordinary differential equation having an additive random term into a linear partial differential equation (the continuity equation) for the corresponding probability density function, this partial differential equation cannot usually be solved analytically. See Refs. 1 and 2 for examples using this approach.

[3] Special consideration is required to define the physical significance of sample functions produced through numerical simulations of stochastic differential equations having a white noise source term since such equations are meaningless in the sample function or mean-square sense. The reader is referred to Ref. 8 for a discussion of the proper mathematical framework of these calculations.

perturbation methods[9] often neglect the nonlinearity in zeroth order, reducing the original problem to a chain of linear equations for the higher-order responses. Another linearization method, which originated in the engineering literature, is called equivalent or statistical linearization.[12] Here, the nonlinear equation is replaced by a linear one in such a way that the time-averaged mean-square error due to replacement is minimized. This method has been studied in detail by Budgor and coworkers[13–15] and has been shown to yield good results for the *equilibrium* properties of the stochastically forced Duffing oscillator.[13]

In this paper, we present a new linearization approximation that we call the piecewise optimal linearization (POL) method. As in statistical linearization, the nonlinear equation is replaced by a linear one that is solved exactly. However, the method is explicitly designed to yield approximations for the time dependence of the statistical properties of the nonlinear stochastic process rather than just the equilibrium, $t \to \infty$, limit. This is achieved by partitioning the range of the independent variable into intervals and then approximating the nonlinear equation by different linear equations in different intervals. Statistical linearization is included as a special case of the method in which only one time interval of infinite length is chosen and the nonlinear equation is approximated by a single linear equation.

Clearly, the transient behavior of a stochastic process can be long lived and will, in fact, be the most important aspect in a stochastic treatment of, say, relaxation problems.[4] However, most previous applications of approximate methods have been concerned only with stationary properties of the random process. For our initial investigation, presented here, we treat the dynamic behavior of a specific nonlinear stochastic process. We present, apparently for the first time, exact results for several time-dependent moments of this simple example and compare these to results obtained from statistical linearization and the POL method. These calculations demonstrate that (1) statistical linearization approximations for the time-dependent moments can be very poor in time regimes far from equilibrium and (2) the POL method represents a significant improvement over the method of statistical linearization in these time regimes. In a future publication, we will extend the POL method to coupled nonlinear generalized Langevin equations and will compare the accuracy and computational efficiency of the POL method to the method of classical stochastic trajectories.[21]

The plan of the paper is as follows. In Section 2, we consider a model nonlinear differential equation with a random driving term and rigorously convert this equation into a hierarchy of coupled differential equations for the moments of the solution distribution function. Numerically exact

results for the first four moments of this distribution are presented. In Section 3, we review the application of the method of statistical linearization to the model nonlinear problem and present the statistical linearization estimates for the moments. The POL method for this problem is outlined and the POL estimates are compared to the exact results in Section 4. We discuss the results in Section 5.

2. MODEL PROBLEM AND EXACT STATISTICS

The example considered here is a stochastically forced Bernoulli oscillator,

$$\dot{X}(t) + \beta X(t) + \lambda X(t)^3 = F(t), \qquad X(0) = X_0 \tag{1}$$

where $F(t)$ is a delta-correlated Gaussian random force satisfying

$$\langle F(t) \rangle = 0 \tag{2}$$

$$\langle F(t)F(t') \rangle = 2D\delta(t - t') \tag{3}$$

with the brackets $\langle\ \rangle$ signifying an average over the distribution of the random force. As indicated in Eq. (1), we assume that the initial value $X(0)$ is a deterministic quantity uncorrelated with the random force. The constant D in Eq. (3), which is related to the bath temperature in many problems, must clearly be positive. Also, the constants λ in Eq. (1) must be positive to guarantee global stability of the solution process. The constant β, however, could be either positive or negative. In these calculations, we restrict β to be positive so that the potential

$$V(X) = (1/2)\beta X^2 + (1/4)\lambda X^4 \tag{4}$$

has a single minimum. This choice is convenient because one expects solutions of Eq. (1) with $\beta > 0$ to approximate the behavior of the Duffing oscillator [with a potential proportional to $V(X)$] in the limit of large damping.[16] Negative values of β lead to a double-minimum potential. Previous work on the Brownian motion in double-well potentials[17] has demonstrated that statistical linearization estimates of the (stationary) covariance function can be very poor owing to strong non-Gaussian characteristics of the exact distribution function. While we expect that, in many interesting problems, these non-Gaussian deviations build as a function of time[18] and that the POL method may still give good estimates of statistical properties at short times, we will defer consideration of this interesting but more difficult problem.

In the limit $D = 0$, Eq. (1) is deterministic and can be easily solved in closed form. Also, in the limit $\lambda = 0$, the stochastic differential equation is linear and one can simply express all statistical properties of the solution

process in terms of those of $F(t)$. In particular, for $\lambda = 0$ and $F(t)$ a Gaussian random function, the solution probability density function $f(X,t)$ for Eq. (1) is also Gaussian. This follows from the fact that the solution process is then a linear transformation of $F(t)$ and the fundamental result that the Gaussian character of a random variable is preserved under linear transformation. However, for $\lambda > 0$, this conclusion does not hold and we turn instead to the Fokker–Planck equation associated with Eqs. (1)–(3):

$$\frac{\partial f(X,t)}{\partial t} = \frac{\partial}{\partial X}\left[(\beta X + \lambda X^3) f(X,t)\right] + D\frac{\partial^2}{\partial X^2} f(X,t) \tag{5}$$

$$f(X,0) = \delta(X - X_0) \tag{6}$$

The presence of the λX^3 term in Eq. (5) precludes a closed-form analytic solution for $f(X,t)$ except at equilibrium, $t = \infty$. This stationary solution, found by equating the right-hand side of Eq. (5) to zero, is

$$f(X,\infty) = \eta \exp\left(-\frac{\beta}{2D} X^2\right)\exp\left(-\frac{\lambda}{4D} X^4\right) \tag{7}$$

where η is a normalization constant. The non-Gaussian characteristics of this exact distribution function for $\beta > 0$ and $\beta < 0$ are apparent in Eq. (7).

Although Eq. (5) cannot be readily solved for $f(X,t)$, it follows from Eqs. (5)–(6) that the moments of $f(X,t)$ defined by

$$\langle X(t)^n \rangle = \int_{-\infty}^{\infty} dX\, X^n f(X,t), \qquad n = 0, 1, 2, \ldots \tag{8}$$

satisfy the infinite set of coupled linear differential equations

$$\frac{d}{dt}\langle X^n \rangle = Dn(n-1)\langle X^{n-2} \rangle - n\beta \langle X^n \rangle - n\lambda \langle X^{n+2} \rangle, \qquad n = 0, 1, \ldots \tag{9}$$

$$\langle X^n(0) \rangle = X_0^n \tag{10}$$

These moment equations are rigorously equivalent to the original nonlinear equation but, for a numerical solution, the coupled equations must be truncated at some finite order. It is well known[19] that an arbitrary truncation of such equations does not even necessarily preserve the moment properties of the exact solution, e.g., that the variance must be positive at all times. However, one expects that if the finite set of equations is made large enough, then the calculated moments should converge to the exact moments. As a method of checking the accuracy of these results, the equilibrium moments can be determined by numerical quadrature using the exact stationary distribution, Eq. (7), and compared to the numerical results from Eq. (9) at large t.

We have carried out this program for Eqs. (9)–(10) with $\beta = \lambda = 1$,

Table I. Exact Moments, $\langle X(t)^n \rangle$, Calculated from Eq. (9) with $\beta = \lambda = 1$, $D = 1/2$, $X_0 = 2$

t	$\langle X \rangle$	$\langle X^2 \rangle$	$\langle X^3 \rangle$	$\langle X^4 \rangle$
0.02	1.82177	3.33486	6.1336	11.3339
0.06	1.55776	2.46258	3.9475	6.4119
0.10	1.36741	1.92020	2.7619	4.0603
0.14	1.22069	1.55316	2.0470	2.7821
0.18	1.10232	1.29024	1.5834	2.0216
0.22	1.00364	1.09422	1.2663	1.5376
0.26	0.919331	0.943742	1.0403	1.2129
0.30	0.845936	0.825710	0.87386	0.98580
0.34	0.781098	0.731607	0.74784	0.82130
0.38	0.723154	0.655651	0.65007	0.69870
0.42	0.670895	0.593765	0.57255	0.60514
0.46	0.623418	0.542981	0.50985	0.53231
0.50	0.580033	0.501078	0.45818	0.47471
0.75	0.375405	0.359431	0.26650	0.29399
1.0	0.245509	0.312949	0.16984	0.23813
1.5	0.105556	0.292225	0.07249	0.21351
2.0	0.045420	0.289897	0.03118	0.21075
3.0	0.008410	0.289606	0.00577	0.21040
4.0	0.001557	0.289602	0.00107	0.21040

$D = 1/2$, $X_0 = 2$, and integrated the moment equations to obtain accurate values of the first four moments out to $t = 4$. Up to 200 coupled equations were used in each of the separate calculations for the even- and odd-order moments to ensure results converging to at least six significant figures for the first and second moments and at least five significant figures for the third and fourth moments. The first four moments as a function of time are presented for reference in Table I. In the next two sections, we compare these results to the statistical linearization and POL method predictions.

3. STATISTICAL LINEARIZATION

To apply the method of statistical linearization to Eq. (1), we rewrite it in the form

$$\dot{X} + \tilde{\beta}X + \Delta(X) = F(t) \tag{11}$$

where Δ is the error

$$\Delta(X) = (\beta - \tilde{\beta})X + \lambda X^3 \tag{12}$$

and $\tilde{\beta}$ is a constant to be determined so that Δ is, in some sense, small. The measure of this error is customarily taken to be the time average of Δ^2 over

the whole time domain

$$E = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Delta(X)^2 \, dt \tag{13}$$

Now, under the assumption that Δ^2 is ergodic, Eq. (13) may be replaced by

$$E = \langle \Delta(X)^2 \rangle_{eq} \tag{14}$$

where $\langle\ \rangle_{eq}$ indicates an average over the equilibrium distribution function, Eq. (7). The parameter $\tilde{\beta}$ in Eq. (1) is now chosen so that E is minimized. Carrying out this minimization for Eq. (14) gives

$$\tilde{\beta} = \beta + \lambda \langle X^4 \rangle_{eq} / \langle X^2 \rangle_{eq} \tag{15}$$

With the neglect of Δ in Eq. (1), the resulting linear equation can be solved analytically, and since F is Gaussian, so will be the approximate probability function $\tilde{f}(X,t)$ for the linearized Eq. (11).

In more general situations, the exact stationary probability density function may not be known, either because a Fokker–Planck equation cannot be derived or, even if it can, the equilibrium solution may not be easily expressed in closed form. Therefore it is usual in applications of statistical linearization to replace the averages in Eq. (15) by averages over the equilibrium probability density function obtained from the linearized Eq. (11).[13] With this replacement, Eq. (15) becomes an implicit equation for the optimal $\tilde{\beta}$ which must be solved self-consistently.[4]

We have applied this self-consistent version of statistical linearization to Eq. (1) with the same parameters used in Section 2, $\beta = \lambda = 1$, $D = 1/2$, $X_0 = 2$. The results for the first four moments of $\tilde{f}(X,t)$, expressed as a percentage error from the exact results, are plotted in Fig. 1. It is seen that these errors are very large for $t < 1$ but decrease in magnitude $t > 1$. Thus, although the optimal $\tilde{\beta}$ may represent the best *global* parameter according to Eq. (13), the local, short-time errors can be quite significant, making the statistical linearization approach unreliable for the treatment of relaxation processes. The relatively small errors occurring near equilibrium are, of course, to be expected from Eq. (15). These numerical results show that statistical linearization should be regarded as a distinctly nonuniform approximation method for estimating dynamical statistical properties of nonlinear stochastic processes. In the next section, we show that the POL method reduces these errors and gives a more uniform approximation to these moments.

[4] According to Ref. 13, use of the exact stationary distribution function in Eq. (15) leads to the "modified version of statistical linearization" (modified SLO).

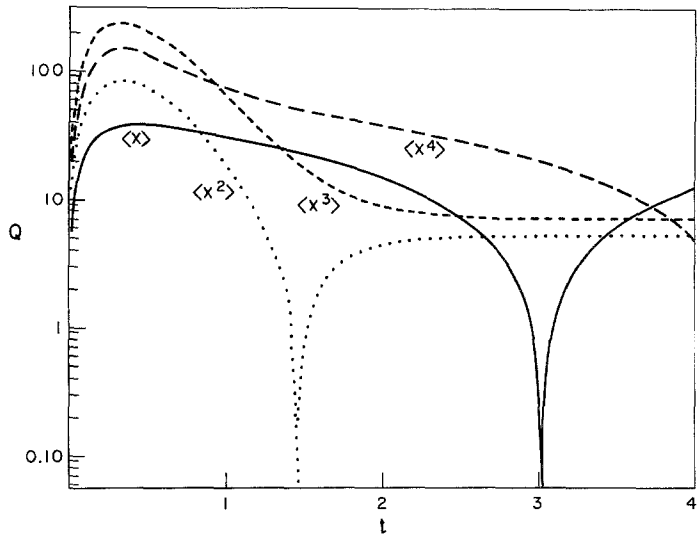

Fig. 1. Statistical linearization errors for the time-dependent moments of Eq. (1) with $\beta = \lambda = 1$, $D = 1/2$, $X_0 = 2$. Q is the percentage error of the nth moment; $Q = [\langle X^n \rangle_{\text{estimated}} - \langle X^n \rangle_{\text{exact}}]/\langle X^n \rangle_{\text{exact}} \times 100\%$, $n = 1$ (——), $n = 2$ (· · ·), $n = 3$ (– – –), $n = 4$ (----). Singularities in this and the following graphs are due to sign changes of Q.

4. PIECEWISE OPTIMAL LINEARIZATION METHOD

To implement the POL method, we first partition the time domain into a set of intervals, each of arbitrary length. Within the jth interval, Eq. (1) is rewritten

$$\dot{X} + \beta_j X + c_j + e_j = F(t), \qquad t_{j-1} \leqslant t \leqslant t_j, \quad t_0 = 0 \tag{16}$$

where β_j and c_j are constants within the interval and the error e_j is

$$e_j = (\beta - \beta_j)X - c_j + \lambda X^3 \tag{17}$$

Again, we want to choose the free parameters β_j and c_j so that a measure of this error is small. Then e_j is neglected in Eq. (16) and the resulting linear equation is solved analytically. As in statistical linearization, the approximate probability density function for the jth interval will be Gaussian and so cannot be expected to adequately reproduce important non-Gaussian characteristics of the exact solution. The added flexibility in the POL method comes from introducing two free parameters into the linearized equation and allowing these to change between intervals.

We choose the parameters β_j and c_j so that

$$\langle e_j \rangle_{t_j} = 0 \tag{18}$$

$$\langle e_j^2 \rangle_{t_j} = \text{minimum} \tag{19}$$

where the subscript t_j indicates that the average is to be taken with respect to the probability density function evaluated at $t = t_j$. The conditions Eq. (18) and (19) are optimal in the sense of the Chebyshev inequality[20]

$$P\{|z - \langle z\rangle| \geqslant \alpha\} \leqslant \sigma_z^2/\alpha^2, \qquad \alpha > 0 \tag{20}$$

which relates the probability of deviations of a random variable z from the mean value $\langle z\rangle$ to its variance σ_z^2. Applied to Eqs. (18) and (19), the Chebyshev inequality assures us that large deviations of the random variable $e_j(t_j)$ from the value zero are rare so that $e_j(t_j)$ is probabilistically small. Carrying out the minimization indicated in Eq. (19) gives the optimal parameters

$$\beta_j = \beta + \lambda\left[\langle X^4\rangle_{t_j} - \langle X^3\rangle_{t_j}\langle X\rangle_{t_j}\right]/\sigma_j^2(t_j) \tag{21}$$

$$c_j = \lambda\left[\langle X^2\rangle_{t_j}\langle X^3\rangle_{t_j} - \langle X\rangle_{t_j}\langle X^4\rangle_{t_j}\right]/\sigma_j^2(t_j) \tag{22}$$

where

$$\sigma_j^2(t_j) = \langle X^2\rangle_{t_j} - \langle X^2\rangle_{t_j} \tag{23}$$

However, since the exact probability density function at $t = t_j$ is unknown, we again impose the self-consistency condition and solve Eqs. (21) and (22) iteratively using the approximate probability density function $f_j(X, t_j)$ obtained from the linearized equation to compute the necessary moments.

The formal solution of Eq. (16) with $e_j = 0$ is

$$X^{(j)}(t) = G_j(t - t_{j-1})X^{(j-1)}(t_{j-1}) + c_j/\beta_j\left[1 - G_j(t - t_{j-1})\right] + \int_{t_{j-1}}^{t} G_j(t - \tau)F(\tau)d\tau, \qquad t_{j-1} \leqslant t \leqslant t_j \tag{24}$$

where

$$G_j(t) = e^{-\beta_j t} \tag{25}$$

Equation (24) involves the initial condition $X^{(j-1)}(t_{j-1})$ so that the solution is (mean-square) continuous at the boundary $t = t_{j-1}$ between the $(j-1)$st and jth intervals. It follows from Eq. (24) with Eqs. (2) and (3) that the mean and variance of $f_j(X, t)$ are

$$\langle X^{(j)}(t)\rangle = G_j(t - t_{j-1})\langle X^{(j-1)}(t_{j-1})\rangle + c_j/\beta_j\left[1 - G_j(t - t_{j-1})\right] \tag{26}$$

$$\sigma_j^2(t) = \left[G_j(t - t_{j-1})\right]^2\sigma_{j-1}^2(t_{j-1}) + D/\beta_j\left\{1 - \left[G_j(t - t_{j-1})\right]^2\right\} \tag{27}$$

Since $f_j(X, t)$ is Gaussian in this example, Eqs. (26) and (27) completely specify $f_j(X, t)$ and are sufficient for calculating the POL approximations of the higher-order moments occurring in Eqs. (21) and (22). In more general situations, where $F(t)$ is not Gaussianly distributed, higher-order moments

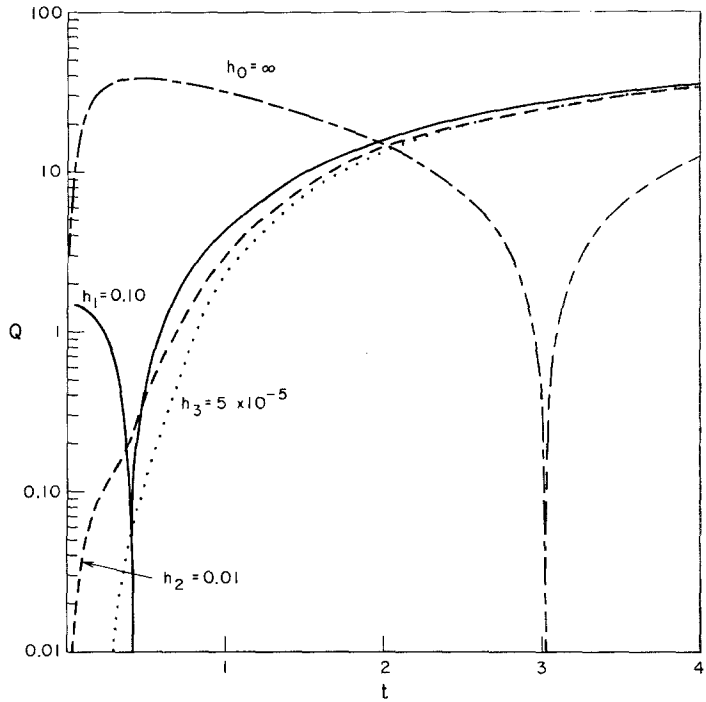

Fig. 2. Integration step size dependence of the POL error for the first moment $\langle X \rangle$ of Eq. (1) with $\beta = \lambda = 1$, $D = 1/2$, $X_0 = 2$. Q is the % error as defined in the caption of Fig. 1; $h_1 = 0.10$ (——), $h_2 = 0.01$ (----), $h_3 = 5 \times 10^{-5}$ (· · ·). The statistical linearization, $h_0 = \infty$, results are indicated as (– – – –).

can be calculated directly from Eq. (24). Propagation formulas similar to Eqs. (26) and (27) would again be obtained.

In the application of the POL method to Eq. (1), we have performed several calculations, integrating each time out to $t = 4$, and keeping the step size constant within each calculation. Figures 2–5 present the POL results for the first four moments of $f_j(X, t_j)$ with three choices of the step size h: $h_1 = 0.10$, $h_2 = 0.01$, and $h_3 = 5 \times 10^{-5}$. Comparison of these results with the moments calculated by statistical linearization ($h_0 = \infty$) clearly shows the dramatic improvement at short times obtained by using the POL method. It is especially interesting that even with the largest step size $h_1 = 0.10$, the percentage errors for the POL results are most often at least an order of magnitude smaller than the statistical linearization errors. Computationally, it should be noted that the 40 iterative cycles necessary for the POL calculation with $h = 0.10$ increases the execution time less than 25% over the statistical linearization calculation. In general, except near the

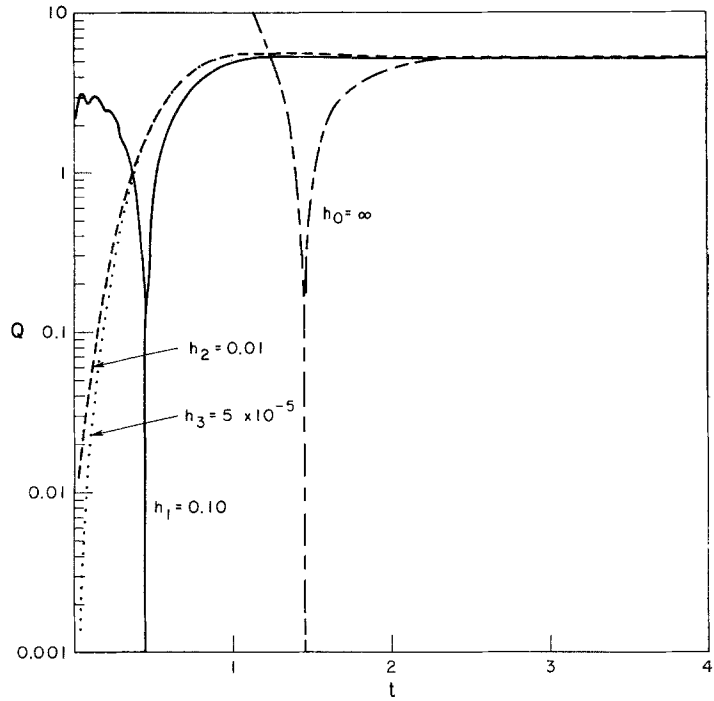

Fig. 3. Same as Fig. 2, except for the second moment $\langle X^2 \rangle$.

points where the error changes sign and near equilibrium, where the errors become nearly equal, the POL errors decrease with decreasing step size. However, the decrease is much more marked going from h_1 to h_2 than from h_2 to h_3; that is, the POL calculations converge rapidly with respect to decreasing step size. Since the linearization scheme can never yield the exact, non-Gaussian distribution function even for infinitesimal step sizes, it is encouraging that a relatively large step size gives such significant improvements over the method of statistical linearization. Asymptotically, the POL moments approach the statistical linearization results. This is to be expected since, as the mean of the stochastic process approaches the equilibrium value $\langle x \rangle = 0$, all odd moments of $f_j(X, \infty)$ vanish and Eq. (21) reduces to Eq. (15), while c_j in Eq. (22) approaches zero. We note, however, that the POL and statistical results are not expected to be the same asymptotically when the mean at equilibrium is not zero.

While we do not present methods for correcting the POL method in this paper, we can easily discuss the ultimate accuracy of the approach by using the exact moments to calculate the POL parameters in Eqs. (21) and (22) and then using the modified density function to calculate approxima-

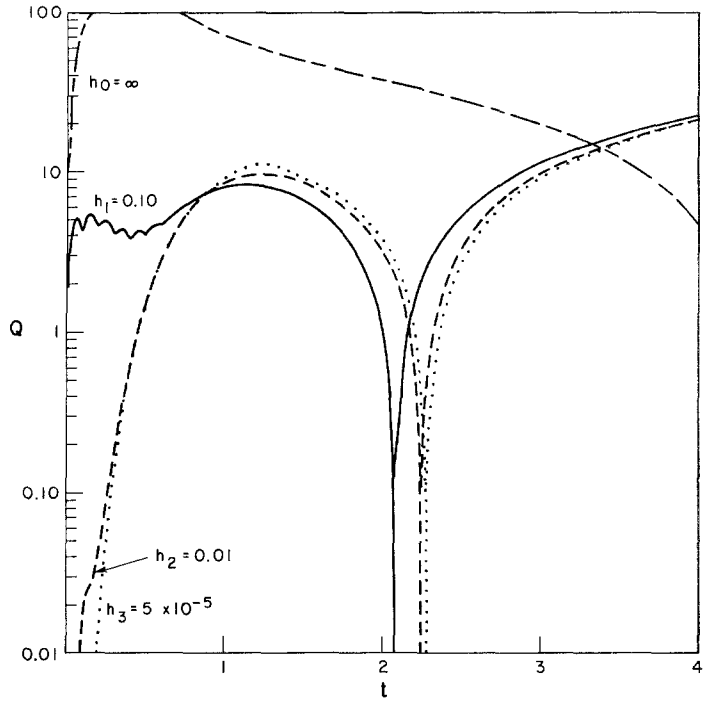

Fig. 4. Same as Fig. 2, except for the third moment $\langle X^3 \rangle$.

tions for these moments. That is, we can answer the question: What is the error introduced by imposing the self-consistent condition of using a density function calculated from the linearized equation to determine the moments appearing in Eqs. (21) and (22)? Budgor *et al.*[13] have previously shown that using the exact equilibrium density function to calculate the necessary quantities in the statistical linearization approximation significantly improves the stationary second-order properties of the Duffing oscillator. Figure 6 displays the modified POL method estimates of the first four moments obtained for a step size $h = 0.01$. Comparison of these results for $h = 0.01$ in Figs. 2–5 (dotted line) shows that the modified POL estimates are consistently more accurate than the self-consistent POL estimates for the first and second moments, especially near equilibrium where the improvement is better than two orders of magnitude. However, for the third and fourth moments, the self-consistent POL estimates are generally better than the modified POL results. This is clearly due to chance cancellation of errors, since for a Gaussian distribution, the higher-order moments are expressed algebraically in terms of the first two mo-

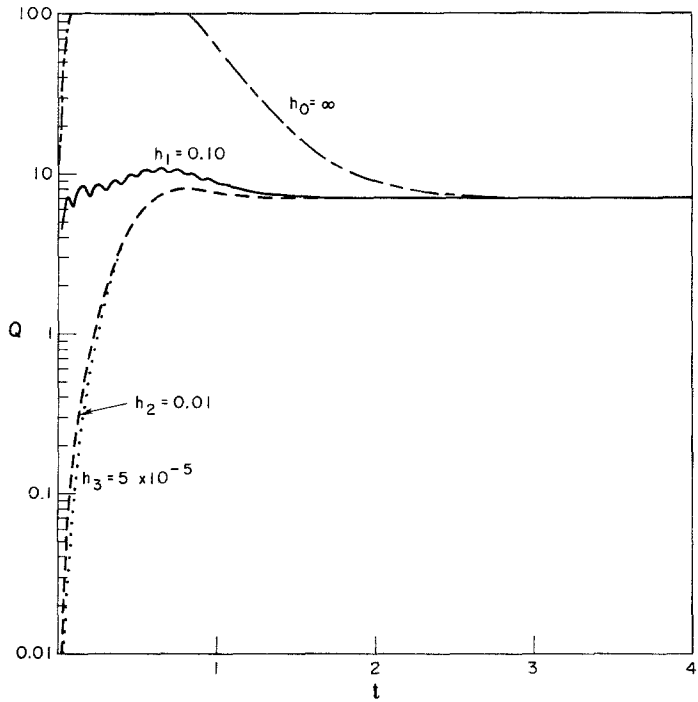

Fig. 5. Same as Fig. 2, except for the fourth moment $\langle X^4 \rangle$.

ments. We conclude that the modified POL method, even if generally applicable, would not guarantee a better set of moments than obtained from the self-consistent POL method.

5. CONCLUSIONS

In this paper we have proposed a piecewise optimal linearization (POL) method for nonlinear stochastic differential equations containing a random inhomogeneity. Through numerical calculations for the time-dependent moments of a simple nonlinear process governed by such an equation, we have verified that this scheme provides significantly better estimates of the moments at short times then does the method of statistical linearization and, at large times, these estimates approach the statistical linearization results. The improvement is typically better than two orders of magnitude for the four moments that we consider. Since the POL method can be easily generalized to coupled nonlinear Langevin equations,[21] we anticipate that it will be a useful technique for approximating the transient

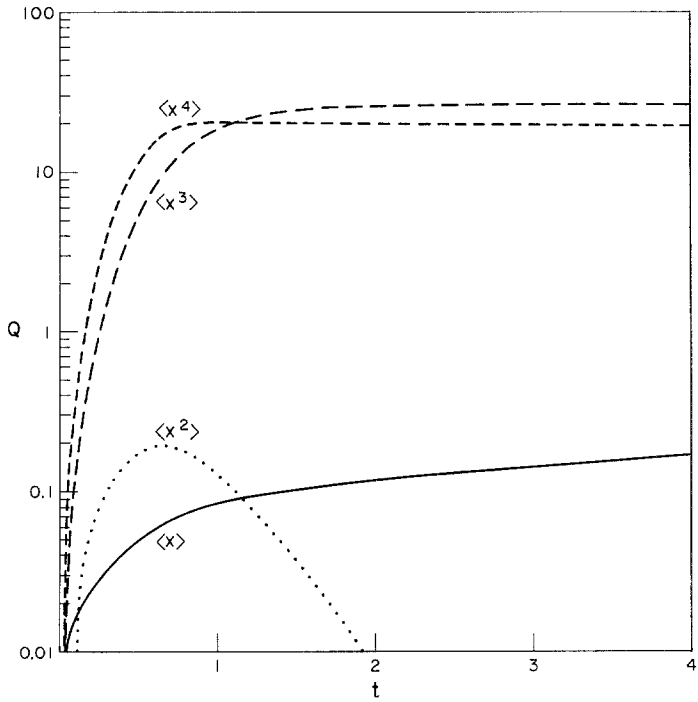

Fig. 6. Same as Fig. 1, except that these are errors for the modified POL method (see text).

statistical behavior in stochastic models of physical systems, especially those driven by Gaussian random forces. For these models, the limiting problem of any linearization method is that the approximate density function for the solution process will also be (jointly) Gaussian. When non-Gaussian characteristics are important,[6,17] neither statistical linearization or the POL method should be expected to yield good estimates of the statistical properties.

REFERENCES

1. N. G. Van Kampen, *Phys. Rep.* **24**:171 (1976).
2. R. F. Fox, *Phys. Rep.* **48**:179 (1978).
3. S. A. Adelman and J. D. Doll, *J. Chem. Phys.* **64**:2375 (1976).
4. M. Shugard, J. C. Tully, and A. Nitzan, *J. Chem. Phys.* **69**:336 (1978).
5. R. Kapral and K. J. Shin, *J. Chem. Phys.* **70**:5623 (1979).
6. R. C. Desai and R. Zwanzig, *J. Stat. Phys.* **19**:1 (1978).
7. F. T. Arecchi, M. Asdente, and A. M. Ricca, *Phys. Rev. A* **14**:383 (1976); F. T. Arecchi and A. M. Ricca, *Phys. Rev. A* **15**:308 (1977).
8. P. A. Ruymgaart and T. T. Soong, *J. Math. Anal. Appl.* **34**:325 (1971).
9. S. H. Crandall, *J. Acoust. Soc. Am.* **35**:1700 (1963).

10. T. K. Caughey, *J. Acoust. Soc. Am.* **35**:1706 (1963).
11. J. D. Doll and D. R. Dion, *Chem. Phys. Lett.* **37**:386 (1976).
12. A. Budgor, *J. Stat. Phys.* **15**:355 (1976).
13. A. Budgor, K. Lindenberg, and K. E. Shuler, *J. Stat. Phys.* **15**:375 (1976).
14. B. J. West, K. Lindenberg, and K. E. Shuler, *J. Stat. Phys.* **18**:217 (1978).
15. A. Budgor and B. J. West, *Phys. Rev. A* **17**:370 (1978).
16. H. A. Kramers, *Physica (The Hague)* 7:284 (1940).
17. K. Matsuo, *J. Stat. Phys.* **18**:535 (1978).
18. S. H. Northrup and J. T. Hynes, *J. Chem. Phys.* **69**:5261 (1978).
19. R. Wilcox and R. Bellman, *J. Math. Anal. Appl.* **32**:532 (1970).
20. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I (Wiley, New York, 1968), p. 233.
21. J. O. Eaves and W. P. Reinhardt, unpublished work.